MAC 2233
PROBLEM SOLVING

Prepared by Gregory Maxwell

Overview:

When solving problems it is helpful to follow the guideline below (**R.E.S.T.**):

- **Read** the problem
  - Basic question: *What do I need to find out?*
  - The aim is to understand what the problem is about. Diagrams and/or tables are helpful most times.
- **Explore** the problem
  - Basic questions: *What information am I given (implicitly or explicitly)? What am I not given?*
  - It is useful to assign a symbol to the quantity you want to find. It is best to try to keep to using only one symbol as much as possible. Diagrams and/or tables also help here.
- **Strategize** (Develop a strategy) & **Solve** using the strategy
  - Basic question: *How can I use what I have to get what I want?*
  - This usually involves the use of formulas or equations. What formula or series of formulas (or equations) relate what you have to what you want?
- **Test** your answer (when possible) to see if it makes sense.
  - Basic question: *Does my answer make sense?*
  - If the answer doesn't make sense, then chances are it is incorrect. Also, *be careful that you answered the question asked*. The problem is not solved until you answer the question asked.

Some Terms and their meanings in word problems:

"When…" or **"How long will it take to …"** refer to **TIME**.

"Where…" or **"How far…"** refer to **DISTANCE** or **POSITION.**

"How fast …" refers to **RATE** or **SPEED** or **VELOCITY.**

"X is no less than Y" or **"X is at least Y"** is written: $\mathbf{X} \geq \mathbf{Y}$ (Read as: **"X is greater than or equal to Y" or "Y is less than or equal to X"**).

"X is no more than Y" or **"X is at most Y"** is written: **XY** (Read as: **"X is less than or equal to Y"** or **"Y is greater than or equal to X."**)

Some Formulas To Remember

If $y = f(x)$, then:

- The **Average Rate of Change** of y from t_1 to t_2 is given by:

$\frac{f(t_2) - f(t_1)}{t_2 - t_1}$ which is the slope of the line joining the points $(t_1, f(t_1))$ and $(t_2, f(t_2))$.

- The **Instantaneous Rate of Change** of y at t_1 is $f'(t_1)$, which is the derivative of $f(t)$ at t_1.

Note: The unit of measurement for rate of change is:

$$\frac{\text{Unit of measurement of f(t)}}{\text{Unit of measurement of t}}$$

If $s(t)$ is the position of a particle at time t, then

- The **Average Velocity** of the paritcle from t_1 to t_2 is given by:

$$V_{AV} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

- The **Instantaneous Velocity** of the particle at t_1 is given by:

$$v(t_1) = s'(t_1)$$

- The **Acceleration** of the particle at t_1 is given by:

$$a(t_1) = v'(t_1) = s''(t_1)$$

which is the first derivative of the velocity function at t_1 and the second derivative of the position function at t_1. That is, the **Acceleration is how fast the velocity is changing** (the rate of change of the velocity).

Important Comments about Position/Displacement, Velocity and Acceleration

- When a particle is at rest (not moving) its velocity is zero (0).
- When a particle has a constant velocity (that is, uniform motion), its acceleration is zero (0).
- If the position/displacement is measured in feet, and time is measured in seconds, then:
  - Velocity is measured in feet per second (ft/s).
  - Acceleration is measured in feet per square second (ft/s^2).

Word Problems Involving Optimization (Maximizing and Minimizing)

In solving word problems involving optimization (that is, in which something is maximized or minimized), it is important to take note of the following information when listing the data from the problem:

- Identify what the question is asking for.
- Identify the quantity to be optimized (that is, maximized or minimized).
- Identify other variables. Oftent there are two variables in addition to the quantity to be optimized.
- Identify the **OBJECTIVE Equation**.
    - The quantity to be optimized is the subject of the objective equation
    - The objective equation is the equation that will be differentiated
- Identify the **CONSTRAINT Equation**.
    - This usually relates the other variables to each other (that is the variables other than the one to be optimized)
    - The constraint equation allows us to express the objective equation in terms of one variable instead of two, and so allows for differentiation of the objective equation.
- Choose one of the variables inthe constraint to be eliminated from the objective equation.
    - Make the chosen variable the subject of the constriaint equation, then substitute for that variable in the objective equation.
- Differentiate the objective equation, the derivative equal to 0, then solve for the variable.
- Complete the problem, being sure to work toward obtaining the information that the problem is asking for.

Comments Regarding Inventory Control Problems

In addition to the information mentioned above, note the following information about **Inventory Control** problems:

Formulas to remember:

[inventory cost] = [ordering/production cost] + [carrying cost]

[ordering/production cost] = [cost per order/production run] × [number of orders]

Suppose x items are produced in each production run then:

- If carrying cost is based on **average** number of items present, then

  [carrying cost] = $\frac{x}{2}$ × [cost per item]

- If carrying cost is based on **maximum** number of items present, then

  [carrying cost] = x × [cost per item]

Note: Assume that the maximum items present is the number produced in each production run and that the same number of items is produced in each production run.

Example 1

A closed rectangular box with square base and a volume of 12 cubic feet is to be constructed using two different types of materials. The top is made of a metal costing \$2 per square foot and the remainder of wood costing \$1 per square foot. Find the dimensions of the box for which the cost of materials is minimized.

Source: Calculus & Its Applications, 10th Ed. by L. Goldstein, D. Lay, & D. Schneider. Exercises 2.5 #16.

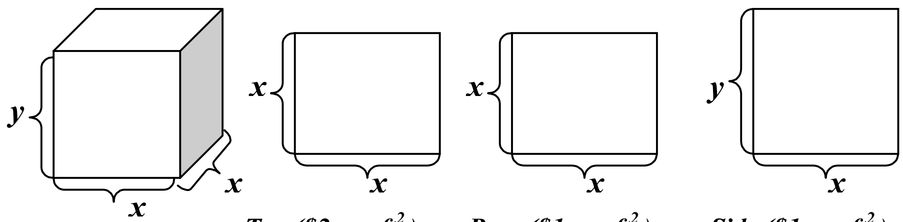
Solution:

We wish to find the dimensions of the box $= x \times x \times y$ (see diagram).

Since the cost is to be minimized, then the cost will be the subject of the objective equation.

Volume of box = length × width × height = $12\,ft^3$

Cost of top = \$2 per ft^2; Cost of other sides = \$1 per ft^2



Top (\$2 per ft²) ***Base (\$1 per ft²)*** ***Side (\$1 per ft²) – 4 sides***

Objective Equation:

Cost = cost of top + cost of base + cost of each side (times 4 for identical sides)

$C = 2x^2 + 1x^2 + 4(1xy) = 3x^2 + 4xy$

Constraint Equation:

Volume = $12\,ft^3 = x \times x \times y$ That is, $x^2 y = 12$

In order to differentiate the objective equation, we must eliminate one of the variables x or y. We eliminate y (because it is easier to replace) by making it the subject of the constraint equation and replacing it in the objective equation:

$x^2 y = 12 \quad \Rightarrow y = \frac{12}{x^2}$ Objective becomes:

$C(x) = 3x^2 + 4x\left(\frac{12}{x^2}\right) = 3x^2 + 48x^{-1}$

$C'(x) = 6x - 48x^{-2}$

Set $C'(x) = 0$:

$6x - 48x^{-2} = 0$ which gives $6x - \frac{48}{x^2} = 0$

$6x = \frac{48}{x^2}$ which gives $x^3 = 8$ So $x = 2$.

To obtain y: $y = \frac{12}{x^2} = \frac{12}{2^2} = \frac{12}{4} = 3$

So the dimensions are $2\,ft \times 2\,ft \times 3\,ft$

Example 2

Find the dimensions of the rectangular garden of greatest area that can be fenced off (all four sides) with 300 meters of fencing.

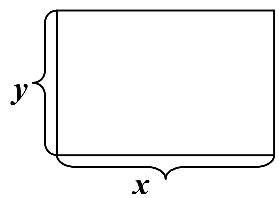
Source: Calculus & Its Applications, 10th Ed. by L. Goldstein, D. Lay, & D. Schneider. Exercises 2.5 #20.

Solution:

We wish to find the dimensions of the garden = $x \times y$ (see diagram).

Since the area is to be maximized, then the area will be the subject of the objective equation.

Amount of fencing = Perimeter = 2× length + 2× width = $300m$



Objective equation: Area = length × width $\Rightarrow A = xy$

Constraint equation: Perimeter = $2x + 2y = 300$ $\Rightarrow x + y = 150$

In order to differentiate the objective equation, we must eliminate one of the variables x or y. We eliminate y (because I prefer x) by making it the subject of the constraint equation and replacing it in the objective equation:

$x + y = 150$ $\Rightarrow y = 150 - x$ Objective equation becomes

$A = x(150 - x) = 150x - x^2$

$A'(x) = 150 - 2x$ and setting $A'(x) = 0$ giving

$150 - 2x = 0$ which gives $2x = 150$ $\Rightarrow x = 75$

To obtain y: $y = 150 - x = 150 - 75 = 75$

So the dimensions of the garden are $75\,ft \times 75\,ft$

Example 3

Find two positive numbers, x and y, whose product is 100 and whose sum is as small as possible.

Source: Calculus & Its Applications, 10th Ed. by L. Goldstein, D. Lay, & D. Schneider. Exercises 2.5 #22.

Solution:

We want x and y such that $xy = 100$ and Sum $S = x + y$ is maximized.

So the objective equation is $S = x + y$ & constraint is $xy = 100$

For variety, we eliminate x by making it the subject of the constraint equation and replacing it in the objective equation:

$xy = 100$ $\Rightarrow x = \frac{100}{y}$ So the objective becomes:

$S(y) = \frac{100}{y} + y = 100y^{-1} + y$

$S'(y) = -100y^{-2} + 1$ which, set to zero, gives

$-100y^{-2} + 1 = 0$ $\Rightarrow 1 = 100y^{-2}$ $\Rightarrow y^2 = 100$ $\Rightarrow y = \pm 10$

We use $y = 10$, since x and y must both be positive (as is stated in the question).

We obtain x by: $x = \frac{100}{y} = \frac{100}{10} = 10$

So the two numbers are $x = 10$ & $y = 10$.

Example 4

The Great American Tire Co. expects to sell 600,000 tires of a particular size and grade during the next year. Sales tend to be roughly the same from month to month. Setting up each production run costs the company $15,000. Carrying costs, based on the average number of tires in storage, amount to $5 per year for one tire.

a. Determine the cost incurred if there are 10 production runs during the year.

b. Find the economic lot size (i.e., the production run size that minimizes the overall cost of producing the tires).

Source: Calculus & Its Applications, 10th Ed. by L. Goldstein, D. Lay, & D. Schneider. Exercises 2.6 #6..

Solution:

Since we will need to determine the number of tires produced in each production, let that be x.

Let r be the number of production runs for the year, since that is not specified and will be needed.

Since the overall cost of production will be minimized, then that will be the subject of the objective equation.

Total number of tires sold for the year = 600,000

Carrying cost is based on **average** number of tires in storage and = $5 per tire for the year.

Assumption: The number of tires produced in each production is the same.

[Total cost] = [production cost] + [carrying cost]

$C = 15000r + 5(\frac{x}{2})$ based on the **average** number of tires in storage

Total number of tires produced = rx; so $rx = 600000$

For part **(a)**:

$r = 10,$ so $x = \frac{600000}{r} = \frac{600000}{10} = 60000$

$C = 15000r + 5(\frac{x}{2}) = 15000(10) + 5(\frac{60000}{2}) = 150000 + 150000 = 300,000$

For part **(b)**

We minimize total cost, so the objective equation is $C = 15000r + 5(\frac{x}{2})$

The constraint is $rx = 600000$

We replace r in the objective equation because we need to find x:

$r = \frac{600000}{x} = 600000x^{-1}$ So objective becomes:

$C(x) = 15000(600000x^{-1}) + \frac{5}{2}x = 9 \times 10^9 x^{-1} + \frac{5}{2}x$

$C'(x) = -9 \times 10^9 x^{-2} + \frac{5}{2}$ which, set to zero, gives:

$-9 \times 10^9 x^{-2} + \frac{5}{2} = 0 \quad \Rightarrow \frac{-9\times 10^9}{x^2} = -\frac{5}{2} \quad \Rightarrow x^2 = 36 \times 10^8$

So $x = 6 \times 10^4 = 60,000,$ which is the required result.

Example 5

A bookstore is attempting to determine the economic order quantity for a popular book. The store sells 8000 copies of this book a year. The store figures that it costs \$40 to process each new order for books. The carrying cost (due primarily to interest payments) is \$2 per book, to be figured on the maximum inventory during an order-reorder period. How many times a year should orders be placed?

Source: Calculus & Its Applications, 10th Ed. by L. Goldstein, D. Lay, & D. Schneider. Exercises 2.6 #8

Solution:

We wish to find the number of orders for the year, r.

The economic order quantity is the number of books per order that costs the least. So, the total cost will be minimized, and so it will be the subject of the objective equation.

Let x be the number of books in each order.

Total number of books ordered for the year = 8000

Cost per order = \$40

Carrying cost = \$2 per book.

Assumptions: The number of books ordered is the same in each order.

The maximum number of books in stock is the order quantity x.

The objective equation is $C = 40r + 2x$, based on the **maximum** number of books in stock

Total number of books = rx; so $rx = 8000$ is the constraint equation.

We solve for r, so we replace x in the objective equation by making it the subject of the constraint and substituting for it in the objective equation:

$rx = 8000 \quad \Rightarrow x = \frac{8000}{r}$ So the objective becomes:

$C(r) = 40r + 2(\frac{8000}{r}) = 40r + 16000r^{-1}$

$C'(r) = 40 - 16000r^{-2}$ which we set to zero:

$40 - 16000r^{-2} = 0 \quad \Rightarrow 40 = \frac{16000}{r^2} \quad \Rightarrow r^2 = \frac{16000}{40} = 400 \quad \Rightarrow r = \pm 20$

We use the positive value of r, since the number of orders is a positve number.

So $r = 20$, which is the desired result.